Normal generation of line bundles on smooth curves

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Outline

1. Gonality, Clifford index and normal generation
   - Gonality
   - Clifford index
   - Normal generation

2. Classification of normally generated line bundles
   - An extremal line bundle with $h^1(\mathcal{L}) \geq 2$
   - Nearly Extremal line bundles
   - Multiple coverings of plane curves

I. Gonality

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<u>Geometric Riemann-Roch Theorem</u>

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Geometric meaning of gonality of X

- $\mathrm{gon}(X) = k$ if and only if in the canonical embedding
  1. any $(k-1)$-points are in general position
  2. but there exists a k-secant $(k-2)$-plane.

Gonality of smooth plane curve of degree d (Namba, 1979)

1. A projection from a point $p \in X$ induces a morphism of degree $d-1$ to $\mathbb{P}^1$. ($\therefore \operatorname{gon}(X) \leq d-1$.)
2. Let D be an effective divisor with $\deg(D) \leq d-1$ and $h^0(D) = 2$.
3. By the geometric RR thm, $h^0(D) = 2 = \deg(D) - \dim\langle D\rangle_K$.
4. By the definition, $h^0(K) - h^0(K(-D)) = \dim\langle D\rangle_K + 1$.
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6. We have the following exact sequence;

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 & & \| & & & & \| & & \\
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7. Any $d-2$ points in $\mathbb{P}^2$ imposes independent conditions on curves of degree $d-3$, i.e., for any divisor D of degree $d-2$, $H^0(\mathcal{O}_{\mathbb{P}^2}(d-3)) \to H^0(\mathcal{O}_D)$ is surjective.
8. Any $d-1$ points in $\mathbb{P}^2$ fails to impose independent conditions on curves of degree $d-3$ if and only if they are collinear.
9. **gon**$(X) = d-1$.

Gonality

Y : a plane curve of degree d with δ number of nodes and $\phi : X \to Y$: a normalization of Y.

- $\mathrm{gon}(X) = d - 2$ if $\delta \leq d - 3$.
  1. $\mathcal{K}_X = \phi^* \mathcal{O}_Y(d-3)(-\Delta)$, Δ :=the set of nodes.
  2. any $d-3$ points in $\mathbb{P}^2$ imposes independent conditions on curves of degree $d-3$ passing through Δ, i.e., for any divisor D of degree $d-3$, $H^0(\mathcal{O}_{\mathbb{P}\neq}(d-3)) \to H^0(\mathcal{O}_D)$ is surjective.
  3. any $d-2$ points in $\mathbb{P}^2$ fails to impose independent conditions on curves of degree $d-3$ if and only if they with one node point are collinear.
- $g(X) = \frac{(d-1)(d-2)}{2} - \delta$ and $\mathrm{gon}(X) = d - 2$.

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- $g(X) = \frac{(d-1)(d-2)}{2} - \delta$ and $\mathrm{gon}(X) = d - 2$.

Brill-Noether Theorem

$\mathrm{gon}(X) \leq [\frac{g+3}{2}]$, and
the equality holds if X is a general curve of genus g.

Clifford index

Clifford Index of X

- Clifford index of a line bundle $\mathcal{L}$;

$$\begin{aligned}\mathrm{Cliff}(\mathcal{L}) &= deg\ \mathcal{L} - 2(h^0(\mathcal{L}) - 1) \\ &= g + 1 - (h^0(\mathcal{L}) + h^0(\mathcal{K}_X\mathcal{L}^{-1})).\end{aligned}$$

- The smaller number is that $\mathcal{L}$ has more sections for its degree.
- $\mathrm{Cliff}(\mathcal{L}) = \mathrm{Cliff}(\mathcal{K}_X\mathcal{L}^{-1})$
- Clifford index of a curve X ;

$$\begin{aligned}\mathrm{Cliff}(X) &= \min\{\mathrm{Cliff}(\mathcal{L}) : h^0(\mathcal{L}) \geq 2,\ \deg(\mathcal{L}) \leq g - 1\} \\ &= \min\{\mathrm{Cliff}(\mathcal{L}) : h^0(\mathcal{L}) \geq 2,\ h^1(\mathcal{L}) \geq 2\}.\end{aligned}$$

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Any reduced irr. non-deg. and linearly normal curve X of degree $d \geq 4r - 7$ in $\mathbb{P}^r (r \geq 2)$ has a $(2r-3)$-secant $(r-2)$-plane.

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Coppens-Martens Theorem

$\mathrm{Cliff}(X) = Gon(X) - 3$ or $\mathrm{Cliff}(X) = Gon(X) - 2$

$\mathrm{Cliff}(X) = k - 2$ if and only if X is a general k-gonal curve

Clifford index of smooth plane curve (Namba, 1979)

1. Let $X \in \mathbb{P}^2$ be a smooth plane curve of degree d.
2. We know that $\text{gon}(X) = d - 1$. ($\text{Cliff}(g^1_{d-1}) = d - 3$)
3. $\text{Cliff}(g^2_d) = d - 4$.
4. Let g^r_{c+2r} be a complete linear system computing the Clifford index of X. Then $r \geq 2$ and $c \leq (d - 4)$.
5. Assume that $r \geq 3$ and $c \leq (d - 5)$. Then g^r_{c+2r} induces a birational morphism by the KKM theorem.

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5. Assume that $r \geq 3$ and $c \leq (d - 5)$. Then g^r_{c+2r} induces a birational morphism by the KKM theorem.
6. If $C' := \varphi_{g^r_{c+2r}}(X)$ is not contained in a hyperquadric of rank ≤ 4, then by the exact sequence

$$0 \to \mathcal{I}_{C'}(2) \to \mathcal{O}_{\mathbb{P}^r}(2) \to \mathcal{O}_{C'}(2) \to 0,$$

we have $h^0(2g^r_{c+2r}) \geq 4r - 2$ and hence
$c \leq \mathit{Cliff}(2g^r_{c+2r}) \leq 2c - 4r + 6 \Rightarrow c + 2r \geq 6r - 6.$

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7. By the theorem of Coppens-Martens, $\exists M$ with $\deg(M) = (2r-3)$ and $\langle M \rangle_{g_e^r} = (r-2)$.
8. Projection from M to $\mathbb{P}^1$ is induced by a linear system $g^1_{\leq(d-2)}$
9. It is a contradiction to the gonality of X.
10. If $C' := \varphi_{g^r_{c+2r}}(X)$ is contained in a hyperquadric of rank ≤ 4, then $g^r_{c+2r} = g^1_e + h^1_{e'}$ and $g = 2c+5$
11. It is a contradiction since $g > 2c+5$.

Gonality and Clifford index

Coppens-Kato Theorem (1990)

If Y : a plane curve with $d \geq 2l + 4 (l \geq 3)$ and $\delta < (l-1)d$
and $\phi : X \to Y$: a normalization of Y,
then $g^1_{2d-9} = g^1_e + D_{2d-9-e}$,
g^1_e: base point free linear system on X cut out
by a pencil of lines in $\mathbb{P}^2$.

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Corollary

If $\phi : X \to Y$: as above, then $\mathrm{Cliff}(X) = d - 4$.

Proof.

Assume that $\mathrm{Cliff}(X) \leq d - 5$ and $\mathcal{L}$ computes the Clifford index. If $h^0(\mathcal{L}) \leq 3$, then $\deg \mathcal{L} \leq (d-5) + 2(h^0(\mathcal{L}) - 1) \leq 2d - 9$. Contradiction to Coppens-Kato Theorem. If $h^0(\mathcal{L}) \geq 4$, then by the secant theorem of Coppens and Martens, we get a contradiction. □

Normal generation

- $\mathcal{L}$ is normally generated if $\mathcal{L}$ is very ample and $\mathrm{Sym}^n H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^{\otimes n})$ is surjective for all $n \geq 0$.
- Noether Theorem
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  The canonical bundle is normally generated unless X is a hyperelliptic.
- Castelnuovo, Mattuck, Mumford and Fujita proved
  any line bundle of degree at least $2g + 1$ is normally generated.
- Lange and Martens showed
  every vey ample line bundle of degree $2g$ is normally generated unless X is a hyperelliptic.
- Arbarello, Cornalba, Griffiths and Harris stated
  A general line bundle of degree $[\frac{3}{2}g + 2]$ or greater defines a projectively normal embedding if X is a sufficiently general curve of genus g.

Normal generation: extremal line bundle

Green-Lazarsfeld Theorem

For any smooth curve X of genus g with a very ample line bundle $\mathcal{L}$, if $\deg(\mathcal{L}) \geq 2g + 1 - 2 \cdot h^1(\mathcal{L}) - \mathrm{Cliff}(X)$, then $\mathcal{L}$ is normally generated.

- Note that the condition $\deg(\mathcal{L}) \geq 2g + 1 - 2 \cdot h^1(\mathcal{L}) - \mathrm{Cliff}(X)$ is equivalent to the assumption that $\mathrm{Cliff}(\mathcal{L}) < \mathrm{Cliff}(X)$.
- A very ample line bundle $\mathcal{L}$ is **extremal** if $\mathrm{Cliff}(\mathcal{L}) = \mathrm{Cliff}(X)$ and $\mathcal{L}$ fails to be normally generated.

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Question;

- Find an extremal line bundle.
- Classify the normally generated line bundles $\mathcal{L}$ with $\mathrm{Cliff}(\mathcal{L}) = \mathrm{Cliff}(X) + \alpha$ for small α.

Theorem (Green-Lazarsfeld (1986))

Let $N(c) = \max\{\frac{(c+2)(c+3)}{2}, 10c+6\}$, $g > N(c)$ *, where* $c = \mathrm{Cliff}(X)$. X *is neither hyperelliptic nor bielliptic.* $\mathcal{L}$ *is an extremal line bundle if and only if* $(X, \mathcal{L})$ *is one of;*

	X	$h^1(\mathcal{L})$	$\phi_{\mathcal{L}}$
I.	*Has a* g^1_{c+2}	0	*Embeds* X *with a 4-secant line*
II.	$c = 2f \geq 4$ X *is a double covering* $\phi : X \to Y \subseteq \mathbb{P}^2$ *of a smooth plane curve* Y *of degree f+2*	1	*Embeds* X *with a 4-secant line*
III.	*as in* II	0	*Embeds* X *with a 6-secant conic but no 4-secant line*

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Theorem (GL Theorem 3)

- *X : a smooth curve of genus g*
- *$\mathcal{L}$: a very ample line bundle on X with*

$$deg\mathcal{L} > \begin{cases} \frac{3g-3}{2}, & \mathcal{L} \text{ is special} \\ \frac{3g-3}{2} + 2, & \mathcal{L} \text{ is nonspecial.} \end{cases} \quad (1)$$

If $\mathcal{L}$ fails to be normally generated, then there is an effective divisor **R** *such that $\varphi_{\mathcal{L}}(R)$ fails to impose independent conditions on quadrics and $\mathcal{A} \simeq \mathcal{L}(-\mathbf{R})$ satisfies*

- $degA \geq \frac{g-1}{2}$,
- $\mathrm{Cliff}(\mathcal{A}) \leq \mathrm{Cliff}(\mathcal{L})$,
- $h^1(\mathcal{A}) \geq h^1(\mathcal{L}) + 2$ *and* $h^0(\mathcal{A}) \geq 2$.

Castelnuovo's genus bound

Let g^r_d: birationally very ample with $d-1 = m(r-1)+\epsilon$,

$$0 \le \epsilon \le r-2$$

$$g \le \pi(d,r) := \frac{m(m-1)}{2}(r-1) + m\epsilon.$$

If $r=2$, then $\pi(d,2) = \frac{(d-1)(d-2)}{2}$.
If $r=3$ and d is even, then $d-1 = 2m+1$, whence $g \le (\frac{d-2}{2})^2$.
If $r=3$ and d is odd, then $d-1=2m$, whence $g \le (\frac{d-1}{2})(\frac{d-3}{2})$.

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Kim-Kim Theorem (2004)

$\pi(d,r) \le \pi(d-2, r-1)$ for $d \ge 3r-2, r \ge 3$

If $d \ge 7$, then $(\frac{d-1}{2})(\frac{d-3}{2}) = \pi(d,3) \le \pi(d-2,2) = \frac{(d-3)(d-4)}{2}$.

KKM Theorem (1990)

Let g^r_{c+2r}: compute the Clifford index c of X, $d \leq g-1, \mathbf{r} \geq \mathbf{3}$.

Then the g^r_{c+2r} is birationally very ample

unless X is hyperelliptic or biellpitic.

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- So, if g^r_d: compute the Clifford index c of X with proper range of $d, r \geq 3$, then the g^r_d is birationally very ample.
- But the genus is big compared to d, r, g^r_d is not birational by the Castelnuovo's genus bound.
- Therefore g^r_d gives a multiple covering of the plane curves or $\mathbb{P}^1$.

Note for $g > \frac{(c+2)(c+3)}{2}$

A smooth plane curve X of degree $d \geq 4$

- $g = \frac{(d-1)(d-2)}{2}$, $c := \mathrm{Cliff}(X) = d - 4$. Therefore, $\mathbf{g = \frac{(c+2)(c+3)}{2}}$.
- $D = H - Z_4$, where Z_4 is 4 collinear points and H is a line section of X.
- $K_X - D$ is very ample since $h^0(D + p + q) = 1$ for any $p, q \in X$.
- $h^0(\mathcal{K}_X - D) = (2g - 2) - (d - 4) - g + 1 + 1$. Therefore $\mathrm{Cliff}(\mathcal{K}_X - D) = d - 4$.
- $h^0(\mathcal{K}_X - D) - h^0(\mathcal{K}_X(-D - Z_4)) = 2 =< D >_{\mathcal{K}_X - D} +1$.
- $\mathcal{K}_X - D$ is an extremal line bundle with $h^1(\mathcal{K}_X - D) = 1$.

1. An extremal line bundle with $h^1(\mathcal{L}) \geq 2$

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Theorem(CKK, 2007)

Assume that

- X is neither hyperelliptic nor bielliptic with $\mathbf{g \geq 2c + 5}$, where g is the genus of X and c is the Clifford index of X.
- A very ample line bundle $\mathcal{M}$ computes the Clifford index of X with $(3c/2) + 3 < \deg\mathcal{M} \leq g - 1$,

then

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then

- $g = 2c + 5$ and $\mathcal{M} = \mathcal{F} \otimes \mathcal{F}'$, where $|\mathcal{F}|, |\mathcal{F}'|$ are pencils of degree $c + 2$,
- $\mathcal{M} \otimes \mathcal{F}$ is an **extremal** line bundle with $h^0(\mathcal{M} \otimes \mathcal{F}) \geq 2$, and $h^1(\mathcal{M} \otimes \mathcal{F}) = 2$.
- $\mathcal{M}$ is half-canonical unless X is a $(c + 2)/2$-fold covering of an elliptic curve.

Theorem (CKK)

If

- $S \subset \mathbb{P}^r$ *be a general* $K3$ *surface with* $Pic(S) =< H >$ *where* H *is a hyperplane section and* $degS = 2r - 2$ *and*
- $X \subset S$ *be a smooth irreducible curve and* $X \in |2H|$,

then

- $\mathcal{O}_X(1)$ *is half-canonical, normally generated, and computes the Clifford index of* X,
- *while there is a base point free pencil* $|\mathcal{F}|$ *such that* $\mathcal{O}_X(1) \otimes \mathcal{F}$ *is an extremal line bundle with* $h^1(\mathcal{O}_X(1) \otimes \mathcal{F}) = 2$.

Proof

1. Since $X \in |2H|$, $\mathcal{O}_X(2)$ is the canonical bundle of X with $g(X) = (2H)^2/2 + 1 = 4r - 3, \deg(X) = (2H)(H) = 4(r-1)$. (Note that $\deg(S) = 2r - 2$.)
2. According to Green's and Lazarsfeld's method of computing the Clifford index of smooth curves on a $K3$ surface, $\mathcal{O}_X(1)$ computes the Clifford index of X. $\therefore \mathrm{Cliff}(X) = 2r - 4$.
3. $g(X) \geq 2c + 5$ and $\deg \mathcal{O}_X(1) \leq g(X) - 1$.
4. The curve X lies on a hyperquadric of rank ≤ 4.
5. The pencil $|\mathcal{F}|$ is induced by the ruling of the hyperquadric.
6. One can prove that $\mathcal{O}_X(1) \otimes \mathcal{F}$ is an extremal line bundle on X with $h^1(\mathcal{L}) = 2$.

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Hence for any $g \equiv 1(\mathrm{mod}\ 4)$, there is a smooth curve X of genus g such that X has a extremal line bundle and a non-extremal line bundle which compute the clifford index of X.

2. Nearly extremal line bundles

Nearly extremal line bundles(Akohori(2005), CKK(in progress))

Assume that $g \geq \max\{\frac{(c+4)(c+3)}{2}, 2c+13\}$, $c := \mathrm{Cliff}(X), f := \frac{c}{2}$ and that X is neither hyperelliptic nor bielliptic. Let $\mathcal{L}$ is an line bundle on X with $\deg(\mathcal{L}) = 2g - 1 - 2h^1(\mathcal{L}) - c$, i.e. $\mathrm{Cliff}(\mathcal{L}) = \mathrm{Cliff}(X) + 1$. Then $\mathcal{L}$ is very ample and fails to be normally generated if and only if the pair $(X, \mathcal{L})$ is the following cases:

(i)	$\phi : C \xrightarrow{m:1} \mathbb{P}^1$, $c+2 \leq m \leq c+3$, $\mathcal{L} \simeq \mathcal{K} - g^1_m - B_{c+3-m} + R_4$, $R_4 \in C_4$, $h^1(\mathcal{L}) = 0$
(ii)	$\phi : C \xrightarrow{2:1} C' \subset \mathbb{P}^2$, $\deg(C') = f+2$, $c = 2f \geq 4$ $\mathcal{L} \simeq \mathcal{K} - \phi^* g^2_{f+2} + R_5$, $R_5 \in C_5$, $h^1(\mathcal{L}) = 0$
(iii)	$\phi : C \xrightarrow{3:1} C' \subset \mathbb{P}^2$, $\deg(C') = \frac{5+c}{3} =: h \geq 3$, $\mathcal{L} \simeq \mathcal{K} - \phi^* g^2_h + R_6$, $R_6 \in C_6$, $h^1(\mathcal{L}) = 0$
(iv)	$\phi : C \xrightarrow{3:1} C' \subset \mathbb{P}^2$, $\deg(C') = \frac{5+c}{3} =: h \geq 4$, $\mathcal{L} \simeq \mathcal{K} - \phi^* g^2_h + R_4$, $R_4 \in C_4$, $h^1(\mathcal{L}) = 1$
(v)	$\phi : C \xrightarrow{\simeq} C' \subset \mathbb{P}^2$, $\mathcal{L} \simeq \mathcal{K} - g^2_{c+5} + R_4$, $R_4 \in C_4$, $h^1(\mathcal{L}) = 1$
(vi)	$\phi : C \xrightarrow{\simeq} C' \subset \mathbb{P}^2$, $\mathcal{L} \simeq \mathcal{K} - g^2_{c+5} + R_6$, $R_6 \in C_6$, $h^1(\mathcal{L}) = 0$

Proof

- If $\mathcal{L}$ is extremal, then by the GL Theorem 3, there exists a line bundle $\mathcal{A}$ with $\mathrm{Cliff}(\mathcal{A}) = \mathrm{Cliff}(X) + 1$ (or $\mathrm{Cliff}(\mathcal{A}) = \mathrm{Cliff}(X)$) and $h^0(\mathcal{A}) \geq 2$ and $h^1(\mathcal{A}) \geq 2$.
- The following Ballico-Keem theorem tells that a special linear series $|\mathcal{K}_X \mathcal{A}^{-1}| = g^r_{2r+c+1}, r \geq 3$ is birationally very ample.
- The condition $g > \max\{\frac{(\mathbf{c+4})(\mathbf{c+3})}{\mathbf{2}}, 2c + 13\}$ gives that any morphism to $\mathbb{P}^r, r \geq 3$ can not be birationally very ample by the Castelnuovo genus bound.
- So $|\mathcal{K}_X \mathcal{A}^{-1}|$ induces a covering morphism to a plane curve or $|\mathcal{K}_X \mathcal{A}^{-1}|$ is a pencil.

Proof

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- So $|\mathcal{K}_X\mathcal{A}^{-1}|$ induces a covering morphism to a plane curve or $|\mathcal{K}_X\mathcal{A}^{-1}|$ is a pencil.

Theorem (BK, 1991)

Let a $|D| = g^r_{2r+c+1}, r \geq 3$ be a special linear series without base point on curve X with Clifford index $c \geq 1$ such that $r(\mathcal{K}_X - D) \geq 1$. Then $|D|$ is birationally very ample.

Extremal line bundles(CKK(in progress))

$g > \max\{\pi(c+4,2), 2c+11\}$ where $c := \mathrm{Cliff}(X)$ and $f := \dfrac{c}{2}$ and that X is neither hyperelliptic nor elliptic-hyperelliptic. Let $\mathcal{L}$ is an line bundle on X with $\deg\mathcal{L} := 2g - 2h^1(\mathcal{L}) - c$. Then $\mathcal{L}$ is very ample and fails to be normally generated if and only if the pair $(X, \mathcal{L})$ is the following cases:

	X	$h^1(\mathcal{L})$	$\mathcal{L}$	conditinos of $R_a \in X_a$
I	$\phi : X \xrightarrow{(c+2):1} \mathbb{P}^1$ ϕ doesn't factor through $\psi : X \overset{2:1}{\to} Y \subset \mathbb{P}^2$ for a smooth Y $\deg Y = f+2$	0	$\mathcal{K}_X - g^1_{c+2} + R_4$ $\langle R_4\rangle_{\mathcal{L}}$: 4-secant line	$\deg(F, R_4) \le 1$, $\forall F \in g^1_{c+2}$
II.	$\phi : X \xrightarrow{2:1} Y \subset \mathbb{P}^2$ for a smooth Y with $\deg Y = f+2$	1	$\mathcal{K}_X - \phi^* g^2_f + R_4$ $\langle R_4\rangle_{\mathcal{L}}$: 4-secant line	$R_4 \le \phi^*(H), H \in \lvert\mathcal{O}_Y(1)\rvert$ $\deg(\phi^*(Q), R_4) \le 1$, $\forall H \in \lvert\mathcal{O}_Y(1)\rvert, Q \le H$
III. (1)	As in II	0	$\mathcal{K}_X - \phi^* g^2_f + R_6$ $\phi_{\mathcal{L}}(R_6) \subset \Omega$ for an irreducible conic $\tilde{\Omega}$ in the plane $\langle R_6\rangle_{\mathcal{L}}$	$\deg(\phi^*(Q), R_6) \le 1, \forall Q \in Y$ $\deg(\phi^*(H), R_6) \le 2$, $\forall H \in \lvert\mathcal{O}_Y(1)\rvert$ $R_6 \le \phi^*(\Omega.Y)$ $\Omega := (\phi \circ \phi_{\mathcal{L}}^{-1})(\tilde{\Omega}) \in \lvert\mathcal{O}_{\mathbb{P}^2}(2)\rvert$
III. (2)	As in II	0	$\mathcal{K}_X - \phi^* g^2_f + R_6$ $\phi_{\mathcal{L}}(R_6) \subset \tilde{L}_1 \cup \tilde{L}_2$ as a scheme in the plane $\langle R_6\rangle_{\mathcal{L}}$	$\deg(\phi^*(Q), R_6) \le 1, \forall Q \in Y$ $(\phi \circ \phi_{\mathcal{L}}^{-1})(\tilde{L}_i) = L_i$: line $\deg(\phi^*(H_i), R_6) = 3$ $\forall H_i := L_i.Y, i = 1, 2$

3. Multiple coverings of plane curves with small number of double points

In this talk, I just deal with the multiple covering of smooth plane curves.

Theorem(Multiple coverings of smooth plane curves)

X : a simple n-fold covering $\phi : X \to Y$ for a smooth plane curve Y of degree d with $g(X) > n(g(Y)) + n(n-1)d + 4n^2(n-1)$.
$\mathcal{L}$: a line bundle, $\deg\mathcal{L} \geq 2g - 2h^1(X,\mathcal{L}) - \mathrm{Cliff}(X) - (n-2)$.
Then, $\mathcal{L}$ is very ample and fails to be normally generated if and only if $\mathcal{L}$ corresponds to one of the cases in the following table.

Theorem(Multiple coverings of smooth plane curves)

X : a simple n-fold covering $\phi : X \to Y$ for a smooth plane curve Y of degree d with $\mathbf{g(X) > n(g(Y)) + n(n-1)d + 4n^2(n-1)}$.
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Then, $\mathcal{L}$ is very ample and fails to be normally generated if and only if $\mathcal{L}$ corresponds to one of the cases in the following table.

Remark

- If $\deg(Y)$ is large compared with n, the genus bound $g > n(g(Y)) + n(n-1)d + 4n^2(n-1)$ is not so restrictive by Riemann-Hurwitz theorem which tells that $g \geq n(g(Y)) - (n-1)$.
- Theorem explores necessary and sufficient conditions for the failure of normal generation of a very ample line bundle $\mathcal{L}$ with $\mathrm{Cliff}(\mathcal{L}) \leq \mathrm{Cliff}(X) + (n-2)$, i.e., $\deg\mathcal{L} \geq 2g - 2h^1(\mathcal{L}) - \mathrm{Cliff}(X) - (n-2)$.

	description for $\mathcal{L}$	$h^1(X,\mathcal{L})$; $\mathrm{Cliff}(\mathcal{L})$	conditions of R_a
I	$\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_4$, $\dim \langle R_4 \rangle_{\mathcal{L}} = 1$	1; $\mathrm{Cliff}(X) + (n-2)$	$R_4 \le \phi^* H$ for some $H \in g^2_d$; $\deg(R_4, \Phi^{-1}(Q)) \le 1$ for any $Q \in \psi(Y) \subset \mathbb{P}^2$
II	$\mathcal{L} \simeq \mathcal{K}_X - (\phi^* g^1_{d-1} + B) + R_4$, $\dim \langle R_4 \rangle_{\mathcal{L}} = 1$; B is a base locus of $\mathcal{K}_X \otimes \mathcal{L}^{-1}(R_4)$	0; $\mathrm{Cliff}(X) + k$, $0 \le k \le n-2$, $k := \deg(B)$	$g^1_{d-1} = g^2_d(-Q)$ for some $Q \in Y$; $\deg(R_4, \phi^*(H-Q)) \le 1$ for any $H \in g^2_d$ with $H \ge Q$; $R_4 \not\le \phi^*(H))$ if $B \le \phi^*(Q)$ and $\deg(\phi^*(Q) - B) \le 2$; $\deg B \le n-2, \deg(B, R_4) = 0$
III	$\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_6$, $R_6 \le \varphi_{\mathcal{L}}(X) \cap \tilde{\Omega}$ for an irreducible conic $\tilde{\Omega}$ in the plane $\langle R_6 \rangle_{\mathcal{L}}$; $\varphi_{\mathcal{L}}(X)$ has no trisecant line	0; $\mathrm{Cliff}(X) + (n-2)$	$\deg(R_6, \phi^{-1}(Q)) \le 1$ for any $Q \in Y$; $\deg(R_6, \phi^*(H)) \le 2$ for any $H \in g^2_d$; $R_6 \le \phi^*(\Omega)$ for some $\Omega \in \lvert 2g^2_d \rvert$ with $\Omega \ne H_1 + H_2$ for any $H_i \in g^2_d$
IV	$\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_6$, $R_6 = R_3^{(1)} + R_3^{(2)}$, $R_3^{(i)} := (R_6, \varphi_{\mathcal{L}}(X) \cap L_i)$, L_i : a line in the plane $\langle R_6 \rangle_{\mathcal{L}}$	0; $\mathrm{Cliff}(X) + (n-2)$	$\deg(R_6, \phi^{-1}(Q)) \le 1$ for any $Q \in Y$; $R_6 = R_3^{(1)} + R_3^{(2)}$ with $R_3^{(i)} \le \phi^*(H_i)$ for some $H_i \in g^2_d$

	description for $\mathcal{L}$	$h^1(X, \mathcal{L})$; $\mathrm{Cliff}(\mathcal{L})$	conditions of R_a
I	$\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_4$, $\dim \langle R_4 \rangle_{\mathcal{L}} = 1$	1; $\mathrm{Cliff}(X) + (n-2)$	$R_4 \leq \phi^* H$ for some $H \in g^2_d$; $\deg(R_4, \Phi^{-1}(Q)) \leq 1$ for any $Q \in \psi(Y) \subset \mathbb{P}^2$
II	$\mathcal{L} \simeq \mathcal{K}_X - (\phi^* g^1_{d-1} + B) + R_4$, $\dim \langle R_4 \rangle_{\mathcal{L}} = 1$; B is a base locus of $\mathcal{K}_X \otimes \mathcal{L}^{-1}(R_4)$	0; $\mathrm{Cliff}(X) + k$, $0 \leq k \leq n-2$, $k := \deg(B)$	$g^1_{d-1} = g^2_d(-Q)$ for some $Q \in Y$; $\deg(R_4, \phi^*(H-Q)) \leq 1$ for any $H \in g^2_d$ with $H \geq Q$; $R_4 \not\leq \phi^*(H))$ if $B \leq \phi^*(Q)$ and $\deg(\phi^*(Q) - B) \leq 2$; $\deg B \leq n-2$, $\deg(B, R_4) = 0$
III	$\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_6$, $R_6 \leq \varphi_{\mathcal{L}}(X) \cap \tilde{\Omega}$ for an irreducible conic $\tilde{\Omega}$ in the plane $\langle R_6 \rangle_{\mathcal{L}}$; $\varphi_{\mathcal{L}}(X)$ has no trisecant line	0; $\mathrm{Cliff}(X) + (n-2)$	$\deg(R_6, \phi^{-1}(Q)) \leq 1$ for any $Q \in Y$; $\deg(R_6, \phi^*(H)) \leq 2$ for any $H \in g^2_d$; $R_6 \leq \phi^*(\Omega)$ for some $\Omega \in \lvert 2g^2_d \rvert$ with $\Omega \neq H_1 + H_2$ for any $H_i \in g^2_d$
IV	$\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_6$ $R_6 = R_3^{(1)} + R_3^{(2)}$, $R_3^{(i)} := (R_6, \varphi_{\mathcal{L}}(X) \cap L_i)$ L_i : a line in the plane $\langle R_6 \rangle_{\mathcal{L}}$	0; $\mathrm{Cliff}(X) + (n-2)$	$\deg(R_6, \phi^{-1}(Q)) \leq 1$ for any $Q \in Y$; $R_6 = R_3^{(1)} + R_3^{(2)}$ with $R_3^{(i)} \leq \phi^*(H_i)$ for some $H_i \in g^2_d$

Remark

- This theorem gives not only concrete constructions but also the existence of large family of such nearly extremal line bundles $\mathcal{L}$, since $\mathcal{L}$ can be constructed by choosing divisors R_a on X in the right boxes of the table.

Lemma A

Assume that $\phi : X \to Y$ satisfy the hypotheses in Theorem.
Let $\mathcal{M}$ be a globally generated line bundle
on X with $\deg \mathcal{M} \leq g - 1$ and $h^0(\mathcal{M}) \geq 2$.
If $\mathrm{Cliff}(\mathcal{M}) \leq nd - 4$, then
$\mathcal{M} \simeq \phi^*(g^2_d)$ or $\mathcal{M} \simeq \phi^*(g^2_d)(-Q)$ where $Q \in Y$.
In particular, we obtain $\mathrm{Cliff}(X) = nd - n - 2$,.

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Assume that $\phi : X \to Y$ satisfy the hypotheses in Theorem. Let $\mathcal{M}$ be a globally generated line bundle on X with $\deg\mathcal{M} \leq g - 1$ and $h^0(\mathcal{M}) \geq 2$. If $\mathrm{Cliff}(\mathcal{M}) \leq nd - 4$, then

$$\mathcal{M} \simeq \phi^*(g^2_d) \quad \text{or} \quad \mathcal{M} \simeq \phi^*(g^2_d)(-Q) \quad \text{where } Q \in Y.$$

In particular, we obtain $\mathrm{Cliff}(X) = nd - n - 2$,.

Proof of the Theorem

Step 1. To apply Green-Lazarsfeld Theorem, check the condition (1) in (GL Theorem 3), i.e.,

$$\deg\mathcal{L} > \begin{cases} \frac{3g-3}{2}, & \mathcal{L} \text{ is special} \\ \frac{3g-3}{2} + 2, & \mathcal{L} \text{ is nonspecial.} \end{cases}$$

Step 2. According to GL Theorem, there is a line bundle

$$\mathcal{A} \simeq \mathcal{L}(-R), \quad R > 0,$$

such that both $\mathcal{A}$ and R satisfy all the conditions in that theorem, i.e.,

- R fails to impose independent conditions on quadrics and
- $\deg A \geq \frac{g-1}{2}$,
- $\mathrm{Cliff}(\mathcal{A}) \leq \mathrm{Cliff}(\mathcal{L})$,
- $h^1(\mathcal{A}) \geq h^1(\mathcal{L}) + 2$ and $h^0(\mathcal{A}) \geq 2$.

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- $h^1(\mathcal{A}) \geq h^1(\mathcal{L}) + 2$ and $h^0(\mathcal{A}) \geq 2$.

Step 3. Prove that $\deg A \geq g - 1$. Therefore $\deg \mathcal{K} A^{-1} \leq g - 1$.

Step 4. Apply Lemma A, we get

$$\mathcal{L} \simeq \mathcal{K}_X - \phi^* g_d^2 + R \quad \text{or} \quad \mathcal{L} \simeq \mathcal{K}_X - (\phi^* g_d^2(-(Q)) + B) + R,$$

for some effective divisor R on X which fails to impose independent conditions on quadrics in $\mathbb{P}H^0(\mathcal{L})^*$.

Step 5. Assume $\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R$. Since $\mathrm{Cliff}(\mathcal{A}) \leq \mathrm{Cliff}(\mathcal{L})$, $\mathrm{Cliff}(\mathcal{L}) = nd - 4$. Hence $\deg R = 6 - 2h^1(\mathcal{L})$ by the RR Theorem. The condition $h^1(X, \mathcal{L}) \leq h^1(X, \mathcal{A}) - 2$ forces $h^1(\mathcal{L}) \leq 1$. $\therefore \mathcal{L}$ corresponds to one of the following cases;

Case 1. $\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_4$ with $h^1(\mathcal{L}) = 1$.
Case 2. $\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_6$ with $h^1(\mathcal{L}) = 0$.

Step 5. Assume $\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R$. Since $\mathrm{Cliff}(\mathcal{A}) \leq \mathrm{Cliff}(\mathcal{L})$, $\mathrm{Cliff}(\mathcal{L}) = nd - 4$. Hence $\deg R = 6 - 2h^1(\mathcal{L})$ by the RR Theorem. The condition $h^1(X, \mathcal{L}) \leq h^1(X, \mathcal{A}) - 2$ forces $h^1(\mathcal{L}) \leq 1$. $\therefore$ $\mathcal{L}$ corresponds to one of the following cases;

Case 1. $\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_4$ with $h^1(\mathcal{L}) = 1$.
Case 2. $\mathcal{L} \simeq \mathcal{K}_X - \phi^* g^2_d + R_6$ with $h^1(\mathcal{L}) = 0$.

Step 6. Assume $\mathcal{L} \simeq \mathcal{K}_X - (\phi^* g^2_d(-Q) + B) + R_a$, $R_a \in X^{(a)}$. Since $h^0(\mathcal{K}_X \otimes \mathcal{A}^{-1}) = h^1(\mathcal{A}) = 2$, we have $h^1(\mathcal{L}) = 0$.
Note that $\dim\langle R_a \rangle_{\mathcal{L}} = a - 3$ due to RR Theorem.
Since $\mathrm{Cliff}(\mathcal{A}) \leq \mathrm{Cliff}(\mathcal{L})$ by GL Theorem 3, we have $a \geq 2(a-3) + 2$, i.e., $a \leq 4$.
If $a < 4$, then $\mathcal{L} \simeq \mathcal{K}_X - (\phi^* g^2_d(-Q) + B - P) + (R_a - P)$ for $P \leq R_a$ which is not very ample. Thus we get $a = 4$ and $\deg(B, R_4) = 0$. Therefore $\mathcal{L}$ is

Case 3. $\mathcal{L} \simeq \mathcal{K}_X - (\phi^* g^2_d(-Q) + B) + R_4$ with $h^1(\mathcal{L}) = 0$, $Q \in Y$.

Thank you!!!